

# Ideals in $\mathcal{P}_G$ and $\beta G$

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**Abstract.** For a discrete group  $G$ , we use the natural correspondence between ideals in the Boolean algebra  $\mathcal{P}_G$  of subsets of  $G$  and closed subsets in the Stone-Čech compactification  $\beta G$  as a right topological semigroup to introduce and characterize some new ideals in  $\beta G$ . We show that if a group  $G$  is either countable or Abelian then there are no closed ideals in  $\beta G$  maximal in  $G^*$ ,  $G^* = \beta G \setminus G$ , but this statement does not hold for the group  $S_\kappa$  of all permutations of an infinite cardinal  $\kappa$ . We characterize the minimal closed ideal in  $\beta G$  containing all idempotents of  $G^*$ .

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## 1 Introduction

We recall that a family  $\mathcal{I}$  of subsets of a set  $X$  is an *ideal* in the Boolean algebra  $\mathcal{P}_G$  of all subsets of  $G$  if  $\emptyset \notin \mathcal{I}$  and  $A \in \mathcal{I}$ ,  $B \in \mathcal{I}$ ,  $C \subseteq A$  imply  $A \cup B \in \mathcal{I}$ ,  $C \in \mathcal{I}$ . A family  $\varphi$  of subsets of  $G$  is a *filter* if the family  $\{X \setminus A : A \in \varphi\}$  is an ideal. A filter maximal by the inclusion is called an *ultrafilter*.

For an infinite group  $G$ , an ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  is called *left (right) translation invariant* if  $gA \in \mathcal{I}$  ( $Ag \in \mathcal{I}$ ) for all  $g \in G$ ,  $A \in \mathcal{I}$ . If  $\mathcal{I}$  is left and right translation invariant then  $\mathcal{I}$  is called *translation invariant*. Clearly, each left (right) translation invariant ideal of  $G$  contains the ideal  $\mathcal{F}_G$  of all finite subsets of  $G$ . An ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  is called a *group ideal* if  $\mathcal{F}_G \subseteq \mathcal{I}$  and if  $A \in \mathcal{I}$ ,  $B \in \mathcal{I}$  then  $AB^{-1} \in \mathcal{I}$ .

Now we endow  $G$  with the discrete topology and identify the Stone-Čech compactification of  $G$  with the set of all ultrafilters on  $G$  and denote  $G^* = \beta G \setminus G$ , so  $G^*$  is the set of all free ultrafilters on  $G$ . Then the family  $\{\overline{A} : A \subseteq G\}$ , where  $\overline{A} = \{p \in \beta G : A \in p\}$  forms the base for the topology of  $\beta G$ . Given a filter  $\varphi$  on  $G$ , we denote  $\overline{\varphi} = \bigcap \{\overline{A} : A \in \varphi\}$ , so  $\varphi$  defines the closed subset  $\overline{\varphi}$  of  $\beta G$ , and each non-empty closed subset  $K$  of  $\beta G$  can be defined in this way:  $K = \overline{\varphi}$  where  $\overline{\varphi} = \{A \subseteq G : K \subseteq \overline{A}\}$ .

We use the standard extension [4, Section 4.1] of the multiplication on  $G$  to the semigroup multiplication on  $\beta G$  such that, for each  $p \in \beta G$ , the mapping  $x \mapsto xp$ ,  $x \in \beta G$  is continuous, and for each  $g \in G$ , the mapping,  $x \mapsto gx$ ,  $x \in \beta G$  is continuous. Given two ultrafilters  $p, q \in \beta G$ , we choose  $P \in p$  and, for each  $x \in P$ , pick  $Q_x \in q$ . Then  $\bigcup_{x \in P} xQ_x \in pq$  and the family of all these subsets forms the base of the product  $pq$ .

It follows directly from the definition of the multiplication in  $\beta G$  that  $G^*$ ,  $\overline{G^*G^*}$  are ideals in  $\beta G$ , and  $\overline{G^*}$  is the unique maximal closed ideal in  $\beta G$ . By Theorem 4.44 from [4], the closure  $\overline{K(\beta G)}$  of the minimal ideal  $K(G)$  of  $\beta G$  is an ideal, so  $\overline{K(\beta G)}$  is the

smallest closed ideal in  $\beta G$ . For the structure of  $\overline{K(\beta G)}$  and some other ideals in  $\beta G$  see [4, Sections 4,6].

For an ideal  $\mathcal{I}$  in  $\mathcal{P}_G$ , we put

$$\mathcal{I}^\wedge = \{p \in \beta G : p \in G \setminus A \text{ for each } A \in \mathcal{I}\},$$

and use the following observations:

- $\mathcal{I}$  is left translation invariant if and only if  $\mathcal{I}^\wedge$  is a left ideal of the semigroup  $\beta G$  ;
- $\mathcal{I}$  is right translation invariant if and only if  $(\mathcal{I}^\wedge)G \subseteq \mathcal{I}^\wedge$ .

We use also the inverse to  $^\wedge$  mapping  $^\vee$ . For a closed subset  $K$  of  $\beta G$ , we take a filter  $\varphi$  on  $G$  such that  $K = \overline{\varphi}$  and put

$$K^\vee = \{G \setminus A : A \in \varphi\}.$$

In section 2, we use a classification of subsets of a group by their size to define some special ideals in  $\mathcal{P}_G$ . In section 3, we study ideals of  $\beta G$  between  $\overline{G^*G^*}$  and  $G^*$ . In section 4, we study ideals between  $\overline{K(\beta G)}$  and  $\overline{G^*G^*}$  and characterize the minimal closed ideal in  $\beta G$  containing all idempotents of  $G^*$ .

## 2 Diversity of subsets of a group

In what follows, all group are supposed to be infinite. Let  $G$  be a group with the identity  $e$ . We say that a subset  $A$  of  $G$  is

- *large* if  $G = FA$  for some  $F \in \mathcal{F}_G$ ;
- *small* if  $L \setminus A$  is large for every large subset  $L$ ;
- *thin* if  $gA \cap A$  is finite for each  $g \in G \setminus \{e\}$ ;
- *n-thin*,  $n \in \mathbb{N}$  if, for every distinct elements  $g_0, \dots, g_n \in G$ , the set  $g_0A \cap \dots \cap g_nA$  is finite;
- *sparse* if, for every infinite subset  $X$  of  $G$ , there exists a finite subset  $F \subset X$  such that  $\bigcap_{g \in F} gA$  is finite.

All above definitions can be unified with usage the following notion [16]. Given a subset  $A$  of a group  $G$  and an ultrafilter  $p \in G^*$ , we define a  $p$ -*companion* of  $A$  by

$$\Delta_p(A) = A^* \cap Gp = \{gp : g \in G, A \in gp\}.$$

Then the following statement hold [16]:

- $A$  is large if and only if  $\Delta_p(A) \neq \emptyset$  for each  $p \in G^*$ ;
- $A$  is small if and only if, for every  $p \in G^*$  and every  $F \in \mathcal{F}_G$ , we have  $\Delta_p(FA) \neq Gp$ ;
- $A$  is thin if and only if,  $\Delta_p(A) \leq 1$  for every  $p \in G^*$ ;
- $A$  is  $n$ -thin if and only if,  $\Delta_p(A) \leq n$  for every  $p \in G^*$ ;
- $A$  is sparse if and only if,  $\Delta_p(A)$  is finite for each  $p \in G^*$ .

Following [1], we say that a subset  $A$  of  $G$  is *scattered* if, for every infinite subset  $X$  of  $A$ , there is  $p \in X^*$  such that  $\Delta_p(X)$  is finite. Equivalently [1, Theorem 1],  $A$  is scattered if each subset  $\Delta_p(A)$  is discrete in  $G^*$ .

We denote by  $Sm_G$ ,  $Sc_G$ ,  $Sp_G$  the families of all small, scattered and sparse subsets of a group  $G$ . These families are translation invariant ideals in  $\mathcal{P}_G$  (see [16, Proposition 1]), and for every group  $G$ , the following inclusions are strict [16, Proposition 12]

$$Sp_G \subset Sc_G \subset Sm_G.$$

We say that a subset  $A$  of  $G$  is *finitely thin* if  $A$  is  $n$ -thin for some  $n \in \mathbb{N}$ . The family  $FT_G$  of all finitely thin subsets of  $G$  is a translation invariant ideal in  $\mathcal{P}_G$  which contains the ideal  $\langle T_G \rangle$  generated by the family of all thin subsets of  $G$ . By [6, Theorem 1.2] and [14, Theorem 3], if  $G$  is either countable or Abelian and  $|G| < \aleph_\omega$  then  $FT_G = \langle T_G \rangle$ . By [14, Example 3], there exists a group  $G$  of cardinality  $\aleph_\omega$  such that  $\langle T_G \rangle \subset FT_G$ .

Clearly,  $FT_G \subseteq Sp_G$ . In the next section, we show that  $FT_G \subset Sp_G$  for every group  $G$ .

**Theorem 2.1.** *For every group  $G$ , we have  $Sm_G^\wedge = \overline{K(\beta G)}$ .*

This is Theorem 4.40 from [4] in the form given in [10, Theorem 12.5].

**Theorem 2.2.** *For every group  $G$ , the following statements hold:*

(i)  $Sp_G^\wedge = \overline{G^*G^*}$ ;

(ii) *for a subset  $A$  of  $G$ ,  $\overline{G^*G^*} \subset \overline{A}$  if and only if, for any infinite subsets  $X, Y$  of  $G$ , there exist  $x \in X$ ,  $y \in Y$  such that  $xy \in A$ ,  $yx \in A$ .*

The statement (i) is Theorem 10 from [2], (ii) is a recent result [11].

For more delicate classifications of subsets of groups and  $G$ -spaces see [5], [9], [15].

### 3 Between $\overline{G^*G^*}$ and $G^*$

**Theorem 3.1.** *For every group  $G$ , the following statements hold:*

(i) *if  $\mathcal{I}$  is a left translation invariant ideal in  $\mathcal{P}_G$  and  $\mathcal{I} \neq \mathcal{F}_G$  then there exists a left translation invariant ideal  $\mathcal{J}$  in  $\mathcal{P}_G$  such that  $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$  and  $\mathcal{J} \subset Sp_G$ ;*

(ii) *if  $\mathcal{I}$  is a right translation invariant ideal in  $\mathcal{P}_G$  and  $\mathcal{I} \neq \mathcal{F}_G$  then there exists a right translation invariant  $\mathcal{J}$  in  $\mathcal{P}_G$  such that  $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$ ;*

(iii) if  $G$  is either countable or Abelian and  $\mathcal{I}$  is a translation invariant ideal in  $\mathcal{P}_G$  such that  $\mathcal{I} \neq \mathcal{F}_G$  then there exists a translation invariant ideal  $\mathcal{J}$  in  $\mathcal{P}_G$  such that  $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$  and  $\mathcal{J} \subset Sp_G$ ;

*Proof.* We use the following auxiliary statement [8. Example 3]:

(\*) if a countable group  $\Gamma$  acts on a set  $X$  then, for every infinite subset  $A$  of  $X$ , there exists a countable subset  $T \subset A$  such that the set

$$\{x \in T : gx \neq x, gx \in T, g \in G\}$$

is finite.

(i) We suppose that  $G$  is countable, put  $\Gamma = G$ ,  $X = G$  and consider the action of  $G$  on  $G$  by the left shifts. We take an infinite subset  $A \in \mathcal{I}$  and apply (\*) to choose a countable thin subset  $T \subseteq A$ . We partition  $T$  into two infinite subsets  $T = B \cup C$  and denote

$$\mathcal{J} = \{Z \subseteq G : Z \neq \emptyset, Z \subset FB \text{ for some } F \in \mathcal{F}_G\}.$$

Clearly,  $\mathcal{J}$  is a left translation invariant ideal and  $\mathcal{J} \neq \mathcal{F}_G$ . Since  $gT \cap T$  is finite for every  $g \in G \setminus \{e\}$ , we have  $C \notin \mathcal{I}$ .

Hence,  $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$ . By the choice of  $T$ , each subset  $Y \in \mathcal{J}$  is a finite union of thin subsets, so  $\mathcal{J} \subset Sp_G$ .

If  $G$  is an arbitrary infinite group then we take a countable subset  $A \in \mathcal{I}$ , consider the subgroup  $H$  of  $G$  generated by  $A$  and denote by  $\mathcal{I}_H$  the restriction of  $\mathcal{I}$  to  $H$ ,  $\mathcal{I}_H = \{Y \cap H : Y \in \mathcal{I}\}$ . By above paragraph, there exists a left invariant ideal  $\mathcal{J}'$  in  $\mathcal{P}_H$  such that  $\mathcal{F}_H \subset \mathcal{J}' \subset \mathcal{I}_H$ ,  $\mathcal{J}' \subset Sp_H$ . Then we put  $\mathcal{J} = \{Y \subseteq G : Y \neq \emptyset, Y \subseteq FZ, F \in \mathcal{F}_G, Z \in \mathcal{J}'\}$ .

(ii) We repeat the proof of (i) with the action of  $G$  on  $G$  by the right shifts.

(iii) If  $G$  is countable then we put  $\Gamma = G \times G$ , consider the action of  $\Gamma$  on  $G$  defined by  $(g, h)x = g^{-1}xh$  and repeat the proof of (i) in the countable case. If  $G$  is Abelian then we apply (i) directly.  $\square$

**Theorem 3.2.** *For every group  $G$ , the following statements hold:*

(i) if  $L$  is a closed left ideal in  $\beta G$  such that  $L \subset G^*$  then there exists a closed left ideal  $L'$  of  $\beta G$  such that  $L \subset L' \subset G^*$ ,  $\overline{G^*G^*} \subset L'$ ;

(ii) if  $R$  is a closed subset of  $G^*$  such that  $R \neq G^*$  and  $RG \subseteq R$  then there exists a closed subset  $R'$  of  $G^*$  such that  $R \subset R' \subset G^*$ ,  $R'G \subseteq R$ ;

(iii) if  $G$  is either countable or Abelian and  $I$  is a closed ideal in  $\beta G$  such that  $I \subset G^*$  then there exists a closed ideal  $I'$  in  $\beta G$  such that  $I \subset I' \subset G^*$ ,  $\overline{G^*G^*} \subset I'$ .

*Proof.* (i) We put  $\mathcal{I} = L^\vee$ , apply Theorem 3.2 (i) and set  $L' = \mathcal{J}^\vee$ . Then  $L'$  is a left ideal in  $\beta G$  and  $L \subset L' \subset G^*$ . Since  $\mathcal{J} \subset Sp_G$ , by Theorem 2.2, we have  $\overline{G^*G^*} \subset L'$ .

(ii) We put  $\mathcal{I} = R^\vee$ , and note that  $\mathcal{I}$  is right translation invariant. We apply Theorem 3.2(ii) and set  $R' = \mathcal{J}^\wedge$ .

(iii) We put  $\mathcal{I} = I^\vee$ , apply Theorem 3.2 (iii) and set  $I' = \mathcal{J}^\wedge$ . Then  $I'$  is a left ideal in  $\beta G$  and  $I'G \subseteq I'$ . Since  $\mathcal{J} \subset Sp_G$ , we have  $I'G \subseteq I'$  so  $I'$  is a right ideal.

**Remark 3.1.** If  $\mathcal{I}$  is a group ideal in  $\mathcal{P}_G$  then, by [13],  $\mathcal{I}^\wedge$  is an ideal in  $\beta G$ . By [12, Theorem 4], if  $G$  is either countable or Abelian and  $\mathcal{I}$  is a group ideal such that  $\mathcal{I} \neq \mathcal{F}_G$  then there exists a group ideal  $\mathcal{J}$  in  $\mathcal{P}_G$  such that  $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$ . If  $A$  is an infinite subset of  $G$  then the subset  $AA$  is not sparse (put  $X = A^{-1}$  in corresponding definition). It follows that if  $\mathcal{I}$  is a group ideal and  $\mathcal{I} \subseteq Sp_G$  then  $\mathcal{I} = \mathcal{F}_G$ .

For a cardinal  $\kappa$ ,  $S_\kappa$  denotes the group of all permutations of  $\kappa$ .

**Theorem 3.3.** *For every infinite cardinal  $\kappa$ , there exists a closed ideal  $I$  in  $\beta S_\kappa$  such that*

(i)  $S_\kappa^* S_\kappa^* \subset I$ ;

(ii) if  $M$  is a closed ideal in  $\beta S_\kappa$  and  $I \subseteq M \subseteq G^*$  then either  $M = I$  or  $M = S_\kappa^*$

*Proof.* We take an arbitrary closed subset  $X = \{x_i : i < \omega\}$  of  $\kappa$  and define a permutation  $f_i$  of  $\kappa$  by  $f_i(x_{2i}) = x_{2i+1}$ ,  $f_i(x_{2i+1}) = x_{2i}$  and  $f_i(x) = x$  for all  $x \in \kappa \setminus \{x_{2i}, x_{2i+1}\}$ . We put  $T = \{f_i : i < \omega\}$  and denote by  $\mathcal{I}$  the smallest translation invariant ideal in  $\mathcal{P}_{S_\kappa}$  containing  $T$ .

We note that  $|gT \cap T| \leq 1$  for every  $g \in G \setminus \{e\}$ . Hence,  $T$  is thin and  $\mathcal{I} \subseteq Sp_{S_\kappa}$ . To see that  $\mathcal{I} \subset Sp_{S_\kappa}$ , we observe that each element of  $\mathcal{I}$  is a countable subset of  $S_\kappa$ , but there are uncountable thin subsets of  $S_\kappa$ .

We assume that there is a translation invariant ideal  $\mathcal{J}$  in  $\mathcal{P}_{S_\kappa}$  such that  $\mathcal{F}_G \subset \mathcal{J} \subset \mathcal{I}$ . Then there exists a countable subset  $T_1$  of  $T$  such that  $T_1 \in \mathcal{J}$ ,  $T \setminus T_1$  is infinite and  $T \setminus T_1 \notin \mathcal{I}$ . We denote  $T_2 = T \setminus T_1$  and take a partition  $\omega = W_1 \cup W_2$  such that  $T_1 = \{f_i : i \in W_1\}$ ,  $T_2 = \{f_i : i \in W_2\}$ . We fix an arbitrary bijection  $\varphi : W_1 \rightarrow W_2$  and define a permutation  $h$  of  $\kappa$  by the following rule.

If  $x \in \kappa \setminus X$  then  $f(x) = x$ .

If  $x \in X$  then we take  $i < \omega$  such that  $x \in \{x_{2i}, x_{2i+1}\}$ .

If  $i \in W_1$  then we choose  $j \in W_2$  such that  $j = \varphi(i)$  and put  $h(x_{2i}) = x_{2j}$ ,  $h(x_{2i+1}) = x_{2j+1}$ .

If  $i \in W_2$  then we take  $k = \varphi^{-1}(i)$  and put  $h(x_{2i}) = x_{2k}$ ,  $h(x_{2i+1}) = x_{2k+1}$ .

By the construction of  $h$ , we have  $hT_1h = T_2$ . Since  $\mathcal{J}$  is translation invariant, we have  $T_2 \in \mathcal{J}$ ,  $T \in \mathcal{J}$  so  $\mathcal{J} = \mathcal{I}$  contradicting  $\mathcal{J} \subset \mathcal{I}$ .

To conclude the proof, we put  $I = \mathcal{I}^\wedge$ . By the construction of  $\mathcal{I}$ ,  $I$  is a closed ideal in  $\beta S_\kappa$  satisfying (i), (ii).  $\square$

**Remark 3.2.** If  $I$  is a subset of  $\beta G$  such that  $\overline{G^*G^*} \subseteq I$  then  $I$  is an ideal in  $G^*$ . It

follows that between  $\overline{G^*G^*}$  and  $G^*$  there are no maximal closed ideals in  $G^*$ .

**Lemma 3.1.** *Let  $\{A_n : n < \omega\}$  be a family of sparse subsets of a group  $G$ ,  $A = \bigcup_{n < \omega} A_n$ . Then  $A$  is sparse provided that the following two conditions are satisfied :*

(i) *for every  $F \in \mathcal{F}_G$  there exists  $K \in \mathcal{F}_G$  such that  $F(A_i \setminus K) \cap F(A_j \setminus K) = \emptyset$  for all  $i < j < \omega$ ;*

(ii) *for every  $g \in G \setminus \{e\}$ , there exists  $m \in \omega$  such that  $gx \notin A$  for each  $x \in \bigcup_{n > m} A_n$ .*

*Proof.* We take an arbitrary ultrafilter  $p \in G^*$  and prove that  $\Delta_p(A)$  is finite. We split the proof in two cases.

Case  $\Delta_p(A_n) \neq \emptyset$  for some  $n < \omega$ . Since  $A_n$  is sparse, we have  $\Delta_p(A_n) = T_p$  for some  $F \in \mathcal{F}_G$ . We show that  $\Delta_p(A) = \Delta_p(A_n)$ . Clearly,  $\Delta_p(A_n) \subseteq \Delta_p(A)$ . We take an arbitrary  $g \in G \setminus T$ , put  $F = T \cup \{g\}$  and choose  $K$  satisfying (i). Then  $A_n \notin gp$  and  $\bigcup\{A_i : i < \omega, i \neq n\} \in gp$  so  $gp \notin \Delta_p(A)$  and  $\Delta_p(A) \subseteq \Delta_p(A_n)$ .

Case  $\Delta_p(A_n) = \emptyset$  for each  $n < \omega$ . We show that  $|\Delta_p(A)| \leq 1$ . Assume the contrary :  $A \in g_1p$ ,  $A \in g_2p$  for distinct  $g_1g_2 \in G$ . We denote  $g = g_1g_2^{-1}$ ,  $q = g_1p$ . Then  $\Delta_p(A) = \Delta_q(A)$ ,  $A \in q$  and  $g^{-1}A \in q$ . We choose  $m$  satisfying (ii). Since  $\Delta_q(A_n) = \emptyset$  for each  $n < \omega$ , we have  $\bigcup_{n > m} (A_n) \in q$  but  $A \notin gq$  and we get a contradiction with  $g^{-1}A \in q$ .  $\square$

**Theorem 3.4.** *For every group  $G$ , we have  $FT_G \subseteq Sp_G$  so  $\overline{G^*G^*} \subset FT_G^\wedge$ .*

*Proof.* Since  $FT_G \subseteq Sp_G$ , we should find a sparse subset  $A$  of  $G$  which is not  $n$ -thin for each  $n \in \mathbb{N}$ . Passing to a countable subgroup of  $G$ , we suppose that  $G$  itself is countable.

We construct  $A$  in the form  $A = \bigcup_{n < \omega} A_n$  to satisfy the conditions (i), (ii) of Lemma 3.1 and such that  $A_n$  is not  $n$ -thin for each  $n > 0$ . For each  $n < \omega$ , we construct  $A_n$  in the form  $A_n = \bigcup_{i < \omega} K_n x_{ni}$  for some finite  $K_n$ ,  $|K_n| = n + 1$ ,  $e \in K_n$  and some sequence  $(x_{ni})_{i < \omega}$  in  $G$ .

We enumerate  $G = \{g_n : n < \omega\}$ ,  $g_0 = e$  and denote  $F_n = \{g_n, \dots, g_n\}$ . We put  $K_0 = \{e\}$ ,  $g_{00} = e$ . Assume that we have chosen  $K_0, \dots, K_n$  and  $\{x_{00}, x_{01}, \dots, x_{0n}, \dots, x_{n0}, x_{n1}, \dots, x_{nn}\}$ , so that following conditions are satisfied:

- (1)  $\{g_n, \dots, g_n\} \cap K_n K_n^{-1} = \emptyset$ ;
- (2)  $F_n K_m x_{mn} \cap F_n K_m \{x_{m0}, \dots, x_{m \ n-1}\} = \emptyset$ ,  $0 \leq m \leq n$ ;
- (3)  $F_n K_n \{x_{n0}, \dots, x_{nn}\} \cap F_n K_m \{x_{m0}, \dots, x_{mn}\} = \emptyset$ ,  $0 \leq m < n$ ;
- (4)  $F_n K_n x_{ni} \cap F_n K_n x_{nj} = \emptyset$ ,  $0 \leq i < j \leq n$ .

Then we choose  $K_{n+1}$  and

$$\{x_{0 \ n+1}, x_{1 \ n+1}, \dots, x_{n \ n+1}\}, \quad \{x_{n+1 \ 0}, x_{n+1 \ 1}, \dots, x_{n+1 \ n+1}\}$$

to satisfy (1), (2), (3), (4) with  $n + 1$  in place of  $n$ . After  $\omega$  steps, we get the family  $\{A_n : n < \omega\}$ .

We put  $K = K_0\{x_{00}, x_{01}, \dots, x_{0n}\} \cup \dots \cup K_n\{x_{n0}, x_{n1}, \dots, x_{nn}\}$ .

By (2), (3), (4),  $F_n(A_i \setminus K) \cap F_n(A_j \setminus K) = \emptyset$  for all  $i < j < \omega$ . Hence, the condition (i) of Lemma 3.1 is satisfied.

By (1), (3), (4),  $g_n(\cup_{i>n} A_i) \cap A = \emptyset$ , so the condition (ii) is satisfied. By Lemma 3.1,  $A$  is sparse. For every  $n < \omega$ , the subsets  $\{g\{x_{ni} : i < \omega\} : g \in K_n\}$  of  $A_n$  are pairwise disjoint. Since  $|K_n| = n + 1$ ,  $A_n$  is not  $n$ -thin.  $\square$

For subsets  $X, Y$  of a group  $G$ , we say that the product  $XY$  is an  $n$ -stripe if  $|X| = n$ ,  $n \in \mathbb{N}$  and  $|X| = \omega$ . It is easy to see that a subset  $A$  of  $G$  is  $n$ -thin if and only if  $A$  has no  $(n + 1)$ -stripes. Thus,  $p \in FT_G^\wedge$  is and only if each member  $P \in p$  has an  $n$ -stripe for every  $n \in \mathbb{N}$ .

We say that  $XY$  is an  $(n, m)$ -rectangle if  $|X| = n$ ,  $|Y| = m$ ,  $n, m \in \mathbb{N}$ . We say that a subset  $A$  of  $G$  has bounded rectangles if there is  $n \in \mathbb{N}$  such that  $A$  has no  $(n, n)$ -rectangles (and so  $(n, m)$ -rectangles for each  $m > n$ ).

We denote by  $BR_G$  the family of all subsets of  $G$  with bounded rectangles.

**Theorem 3.5.** *For a group  $G$ , the following statements hold:*

(i)  $BR_G$  is a left translation invariant ideal in  $\mathcal{P}_G$  ;

(ii)  $BR_G^\wedge$  is a closed ideal in  $\beta G$  and  $p \in BR_G^\wedge$  if and only if each member  $P \in p$  has an  $(n, n)$ -rectangle for every  $n \in \mathbb{N}$ ;

(iii)  $BP_G \subset FT_G$ .

*Proof.* (i) If  $XY$  is an  $(n, n)$ -rectangle then  $(gX)Y$  and  $X(Yg)$  are  $(n, n)$ -rectangles, so the family  $BP_G$  is translation invariant.

We take  $AB \in BP_G$  and choose  $n \in \mathbb{N}$  such that  $A, B$  have no  $(n, n)$ -rectangles. By the bipartite Ramsey theorem [3, p. 95], there is a natural number  $r$  such that, for every 2-coloring of edges of the complete bipartite graph  $K_{r,r}$ , one can find a monochrome copy of  $K_{n,n}$ . We assume that  $A \cup B$  contains an  $(r, r)$ -rectangle  $XY$ . We define a coloring  $\chi : X \times Y \rightarrow \{0, 1\}$  of the Cartesian product  $X \times Y$  by the rule:  $\chi((x, y)) = 1$  if and only if  $xy \in A$ . By the choice of  $r$ , there exist  $X' \subset X$ ,  $Y' \subset Y$  such that  $|X'| = |Y'| = n$  and  $X' \times Y'$  is monochrome. Then either  $X'Y' \subset A$  or  $X'Y' \subset B$  and we get a contradiction with the choice of  $A$  and  $B$ . Hence,  $BP_G$  is an ideal in  $\mathcal{P}_G$ .

(ii) By (i),  $BP_G^\wedge$  is a left ideal and  $(BP_G^\wedge)G \subseteq BP_G^\wedge$ . Since  $BP_G \subseteq FT_G \subset Sp_G$  and  $Sp_G^\wedge = \overline{G^*G^*}$ , we have  $(BP_G^\wedge)G^* \subseteq BP_G^\wedge$  so  $BP_G^\wedge$  is a right ideal. The second statement of (ii) is evident.

(iii) Passing to subgroups, we suppose that  $G$  is countable and construct  $A \in FT_G \setminus BP_G$  in the form  $A = \bigcup_{n < \omega} X_n Y_n$ ,  $|X_n| = |Y_n| = n + 1$ . We enumerate  $G = \{g_n : n < \omega\}$ ,  $g_0 = e$  and put  $X_0 = Y_0 = \{0\}$ . Suppose that we have chosen  $X_0 Y_0, \dots, X_n Y_n$ . We

choose  $X_{n+1}Y_{n+1}$ ,  $|X_{n+1}| = |Y_{n+1}| = n + 2$  to satisfy the following conditions for each  $i \in \{1, \dots, n + 1\}$ :

$$g_i X_{n+1} Y_{n+1} \cap X_{n+1} Y_{n+1} = \emptyset, \quad g_i X_n Y_n \cap (X_0 Y_0 \cup \dots \cup X_n Y_n) = \emptyset$$

After  $\omega$  steps, we get the desired  $A$ . Indeed,  $X_n Y_n \subset A$  so  $A \notin BP_G$ . By the construction,  $gA \cap A$  is finite for each  $g \in G \setminus \{e\}$ , so  $A$  is thin and  $A \in FT_G$ .  $\square$

## 4 Between $\overline{K(G)}$ and $\overline{G^* G^*}$

Let  $(g_n)_{n \in \omega}$  be an injective sequence in a group  $G$ . The set

$$\{g_{i_1} g_{i_2} \dots g_{i_n} : 0 \leq i_1 < i_2 < \dots < i_n < \omega\}$$

is called an *FP-set*.

Given a sequence  $(b_n)_{n \in \omega}$  in  $G$ , we say that the set

$$\{g_{i_1} g_{i_2} \dots g_{i_n} b_{i_n} : 0 \leq i_1 < i_2 < \dots < i_n < \omega\}$$

is a *piecewise shifted FP-set*.

**Theorem 4.1.** *For a group  $G$ , the following statements hold:*

(i)  $Sc_G^\wedge = cl\{\epsilon p : \epsilon \in G^*, p \in \beta G, \epsilon \epsilon = \epsilon\}$ ;

(ii)  $Sc_G^\wedge$  is an ideal in  $\beta G$  and  $p \in Sc_G^\wedge$  if and only if each member of  $p$  contains a pierwise shifted FP-set;

(iii)  $Sc_G^\wedge$  is the minimal close ideal in  $\beta G$  containing all idempotents of  $G^*$ .

*Proof.* (i) We remind that a subset  $A$  of  $G$  is scattered if and only if, for each  $p \in A^*$ , the subset  $Gp$  is discrete in  $\beta G$ . Hence,  $A$  is not scattered if and only if, there is  $p \in A^*$  such that  $Gp$  is not discrete. On the other hand  $Gp$  is not discrete if and only if  $p = \epsilon p$  for some idempotent  $\epsilon \in G^*$ .

(ii) Since  $Sc_G$  is a left translation invariant,  $Sc_G^\wedge$  is a left ideal in  $\beta G$ . By (i),  $(Sc_G^\wedge)q \subseteq Sc_G^\wedge$  for each  $q \in \beta G$ , so  $Sc_G^\wedge$  is a right ideal.

By [1, Theorem 1], a subset  $A$  is scattered if and only if  $A$  contains no pierwise shifted FP-sets.

(iii) Let  $\mathcal{M}$  denotes the minimal closed ideals of  $\beta G$  containing all idempotents of  $\beta G$ . By (i),  $Sc_G^\wedge \subseteq \mathcal{M}$ . Since  $Sc_G^\wedge$  is a closed ideal, we have  $\mathcal{M} = Sc_G^\wedge$ .  $\square$

**Remark 4.1.** If  $\mathcal{I}$  is a group ideal in  $\mathcal{P}_G$  and  $\mathcal{I} \subseteq Sp_G$  then  $\mathcal{I} = \mathcal{F}_G$  (see Remark 3.1). We can not state the same if  $\mathcal{I} \subseteq Sc_G$ .



Let  $G$  be the direct sum  $\oplus_{\omega} \mathbb{Z}_2$  of  $\omega$  copies of  $\mathbb{Z}_2 = \{0, 1\}$ . For  $g \in G$ , we denote by  $\text{supt}(g)$  the number of non-zero coordinates of  $g$ . We put  $A = \{g \in G : \text{supt}(g) = 1\}$  and consider the minimal group ideal  $\mathcal{I}$  in  $\mathcal{P}_G$  such that  $A \in \mathcal{I}$ . If  $S \in \mathcal{I}$  then there is  $m \in \mathbb{N}$  such that  $\text{supt}(g) \leq m$  for each  $g \in S$ . It follows that  $S$  has no piecewise shifted  $FP$ -sets, so  $S$  is scattered and  $\mathcal{I} \subset Sc_G$ .

The following observation follows directly from the basic properties of multiplication in  $\beta G$ : each right shift is continuous and each left shift on element of  $g$  is continuous.

**Lemma 3.1.** *If  $L$  is a left ideal in  $\beta G$  and  $R$  is a right ideal in  $\beta G$  then  $\overline{LR}$  is an ideal in  $\beta G$ .*

For a group  $G$ , we put  $I_{G,0} = G^*$  and  $I_{G,n+1} = \overline{G^* I_{G,n}}$ . By Lemma 4.1, each  $I_{G,n}$  is an ideal in  $\beta G$ .

Clearly,  $I_{G,n+1} \subseteq I_{G,n}$  so  $I_{G,n} \subseteq \overline{G^* G^*}$  for  $n > 0$ .

**Theorem 4.2.** *For every group  $G$  and  $n \in \omega$ , we have*

$$(i) \quad I_{G,n+1} \subset I_{G,n}$$

$$(ii) \quad Sc_G^\wedge \subset I_{G,n}.$$

*Proof.* (i) We note that  $I_{G,n+1}^\vee = \{A \subseteq G : \Delta_p(A) \text{ is finite for each } p \in I_{G,n}\}$  and apply Theorem 4 from [7] stating that  $I_{G,n}^\vee \subset I_{G,n+1}^\vee$ .

(ii) For  $n = 0$ , this is evident. We take an idempotent  $e \in G^*$ ,  $p \in \beta G$  and assume that  $ep \in I_{G,n-1}$ . Then  $eep \in G^* I_{G,n-1}$ , so  $eep \in I_{G,n}$ . Applying Theorem 4.1, we conclude that  $Se_G^\wedge \subseteq I_{G,n}$ . The strict inclusion follows from (i).  $\square$

For a natural number  $n$ , we denote by  $(G^*)^n$  the product of  $n$  copies of  $n$ . By Lemma 4.1,  $(G^*)^n$  is an ideal in  $\beta G$ . Clearly,  $\overline{(G^*)^{n+1}} \subseteq \overline{(G^*)^n}$  and  $\overline{(G^*)^n} \subseteq I_{G,n}$ .

By analogy with Theorem 4.2, we can prove

**Theorem 4.3.** *For every group  $G$  and  $n \in \omega$ , we have*

$$(i) \quad \overline{(G^*)^{n+1}} \subset \overline{(G^*)^n};$$

$$(ii) \quad Sc_G^\wedge \subset \overline{(G^*)^n}.$$

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